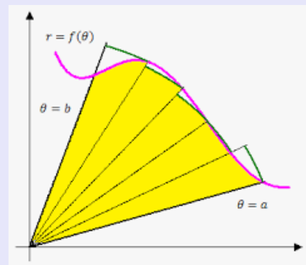
Calculus II

Lecture 13



Feb 19-8:47 AM

Evaluate $\displaystyle\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = 1^{\infty}$ I.F.

Let $y = \left(1+\frac{a}{x}\right)^{bx}$

$\ln y = \ln\left(1+\frac{a}{x}\right)^{bx}$

$\ln y = bx\ln\left(1+\frac{a}{x}\right)$

$e^{\ln y} = y$

$\displaystyle\lim_{x\to\infty}\ln y = \lim_{x\to\infty} bx\ln\left(1+\frac{a}{x}\right) = \infty\cdot 0$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{bx}} = \frac{0}{0}$$

$$= \lim_{x\to\infty}\frac{\frac{\frac{-a}{x^2}}{1+\frac{a}{x}}}{\frac{1}{b}\cdot\frac{-1}{x^2}} = \frac{a}{\frac{1}{b}}\lim_{x\to\infty}\frac{\frac{\frac{1}{x^2}}{1+\frac{a}{x}}}{\frac{1}{x^2}}$$

$$= ab\lim_{x\to\infty}\frac{x^2\cdot\frac{1}{x^2}}{1+\frac{a}{x}} = ab\lim_{x\to\infty}\frac{1}{1+\frac{a}{x}}$$

$$= ab\cdot 1 = \boxed{ab}$$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = e^{ab}$$

Jul 2-8:02 AM

Evaluate $\int_0^{1/2} x\cos\pi x\,dx$

$u=\pi x \rightarrow \frac{u}{\pi}=x$

$du=\pi\,dx$

$x=0 \quad u=0$

$x=\frac{1}{2} \quad u=\frac{\pi}{2}$

$$\int_0^{1/2} x\cos\pi x\,dx=\int_0^{\pi/2}\frac{u}{\pi}\cos u\,\frac{du}{\pi}$$

$$=\frac{1}{\pi^2}\int_0^{\pi/2} u\cos u\,du=\frac{1}{\pi^2}\int_0^{\pi/2} x\cos x\,dx$$

$u=x \quad dV=\cos x\,dx$

$du=dx \quad V=\sin x$

$$\int x\cos x\,dx=x\sin x-\int\sin x\,dx$$

$$=x\sin x+\cos x$$

$$=\frac{1}{\pi^2}\Big[x\sin x+\cos x\Big]_0^{\pi/2}$$

$$=\frac{1}{\pi^2}\left[\frac{\pi}{2}\cdot 1+0-0-1\right]$$

$$=\boxed{\frac{1}{\pi^2}\left[\frac{\pi}{2}-1\right]}$$

Jul 2-8:12 AM

Evaluate $\int_0^{\pi/6}\sqrt{1+\cos 2x}\,dx$

$\cos^2 x=\frac{1+\cos 2x}{2}$

$2\cos^2 x=1+\cos 2x$

$$=\int_0^{\pi/6}\sqrt{2\cos^2 x}\,dx=\sqrt{2}\int_0^{\pi/6}\sqrt{\cos^2 x}\,dx$$

$$=\sqrt{2}\int_0^{\pi/6}\cos x\,dx=\sqrt{2}\sin x\Big|_0^{\pi/6}$$

$\cos x\geq 0$ in QI.

$$=\sqrt{2}\left[\cancel{\sin\frac{\pi}{6}}^{\,\frac{1}{2}}-\cancel{\sin 0}^{\,0}\right]=\boxed{\frac{\sqrt{2}}{2}}$$

Jul 2-8:19 AM

Evaluate $\int_{\pi/6}^{\pi/2} \cot^2 x \, dx$

$1 + \cot^2 x = \csc^2 x$

$\cot^2 x = \csc^2 x - 1$

$\frac{d}{dx}[\cot x] = -\csc^2 x$

$$\int_{\pi/6}^{\pi/2} \cot^2 x \, dx = \int_{\pi/6}^{\pi/2} [\csc^2 x - 1] \, dx$$

$$= (-\cot x - x)\Big|_{\pi/6}^{\pi/2}$$

$\frac{3\pi}{6} - \frac{\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$

$$= -\left[\cot x + x\right]\Big|_{\pi/6}^{\pi/2} = -\left[\cancel{\cot \frac{\pi}{2}}^{\,0} + \frac{\pi}{2} - \cot \frac{\pi}{6} - \frac{\pi}{6}\right]$$

$$= -\left[\frac{\pi}{3} - \sqrt{3}\right]$$

$\cot \frac{\pi}{6} = \cot 30^\circ = \frac{\cos 30^\circ}{\sin 30^\circ} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$

$$= \boxed{\sqrt{3} - \frac{\pi}{3}}$$

Jul 2-8:24 AM

Evaluate $\int \sqrt{5 + 4x - x^2} \, dx$

Hint: Use Completing the Square.

$5 + 4x - x^2 =$

$5 - (x^2 - 4x) =$ $\quad \left[\frac{1}{2}(4)\right]^2 = 4$

$5 - (x^2 - 4x + 4 - 4) =$

$5 - (x^2 - 4x + 4) + 4 =$

$9 - (x-2)^2$

$\int \sqrt{9 - (x-2)^2} \, dx$

$u = x - 2 \quad du = dx$

$= \int \sqrt{9 - u^2} \, du$

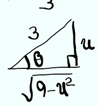
$u = 3\sin\theta$

$\sin\theta = \frac{u}{3}$

$\sqrt{a^2 - x^2} \to x = a\sin\theta$

$\sqrt{a^2 + x^2}$

$\sqrt{x^2 - a^2}$



$= \int \sqrt{9 - 9\sin^2\theta} \cdot 3\cos\theta \, d\theta$

$= 9\int \cos^2\theta \, d\theta = 9\int \frac{1 + \cos 2\theta}{2} \, d\theta$

$= \frac{9}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] = \frac{9}{2}\left[\sin^{-1}\frac{u}{3} + \frac{1}{\cancel{2}} \cdot \cancel{2}\sin\theta\cos\theta\right]$

$= \frac{9}{2}\left[\sin^{-1}\frac{u}{3} + \frac{u}{3} \cdot \frac{\sqrt{9 - u^2}}{3}\right] + C$

Replace u with $x - 2$ $\to$

Jul 2-8:29 AM

Evaluate $\int \frac{x^3+4}{x^2+4}\,dx = \int \left(x + \frac{-4x+4}{x^2+4}\right)dx$

Long Division

$$= \frac{x^2}{2} + \int \frac{-4x+4}{x^2+4}\,dx$$

$$\begin{array}{r} x \\ x^2+4 \overline{\smash{)}\, x^3 \quad +0x^2 \quad +0x \quad +4} \\ -(x^3 \qquad\qquad +4x \quad) \\ \hline -4x \quad +4 \end{array}$$

$$= \frac{x^2}{2} + \int \left(\frac{-4x}{x^2+4} + \frac{4}{x^2+4}\right)dx$$

$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

$$= \frac{x^2}{2} - 2\int \frac{2x}{x^2+4}\,dx + 4\int \frac{1}{x^2+4}\,dx$$

$u = x^2+4$

$du = 2x\,dx$

$\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$

$$= \frac{x^2}{2} - 2\int \frac{1}{u}\,du + 4\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

Jul 2-8:41 AM

Evaluate $\int \frac{x+4}{x^2+2x+5}\,dx = \int \frac{x+1 \quad +3}{x^2+2x+5}\,dx$

$u = x^2+2x+5$

$du = (2x+2)\,dx$

$\frac{du}{2} = (x+1)\,dx$

$\int \frac{1}{u}\cdot\frac{du}{2} = \frac{1}{2}\ln u$

$= \frac{1}{2}\ln(x^2+2x+5)$

$$= \int \frac{x+1}{x^2+2x+5}\,dx + \int \frac{3}{x^2+2x+5}\,dx$$

$u = x^2+2x+5$

$du = (2x+2)\,dx$

$\frac{du}{2} = (x+1)\,dx$

$+\int \frac{3}{x^2+2x+1+4}\,dx$

$+3\int \frac{1}{(x+1)^2+2^2}\,dx$

$+\frac{3}{2}\tan^{-1}\frac{x+1}{2}$

$$= \boxed{\frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}\frac{x+1}{2} + C}$$

note: $x^2+2x+5 = (x+1)^2+4 \geq 4$

Jul 2-8:49 AM

Evaluate $\int \frac{1}{1-\cos x}\,dx$ by using the following

Subs. $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2}{1+t^2}\,dt$.

$$\int \frac{1}{1-\cos x}\,dx = \int \frac{1}{1-\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$

$$= 2\int \frac{1}{1+t^2-(1-t^2)}\,dt = \cancel{2}\int \frac{1}{\cancel{2}t^2}\,dt$$

$$= \int t^{-2}\,dt = \frac{t^{-1}}{-1} + C = \frac{-1}{t} + C$$

$\cos x = \frac{1-t^2}{1+t^2}$

$$\cos x + t^2\cos x = 1 - t^2$$

$$t^2\cos x + t^2 + \cos x - 1 = 0$$

$$t^2(\cos x + 1) = 1 - \cos x$$

$$t^2 = \frac{1-\cos x}{1+\cos x} \to t = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$

Jul 2-8:57 AM

Evaluate $\int_1^{\infty} \frac{1}{(2x+1)^3}\,dx$

$2x+1=0$

$x = \frac{-1}{2}$

No worries

$$\int_1^{\infty} \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{(2x+1)^3}\,dx$$

$u = 2x+1$

$x=1 \to u=3$

$x=t \to u=2t+1$

$$\int_1^{t} \frac{1}{(2x+1)^3}\,dx = \int_3^{t} \frac{1}{u^3}\,\frac{du}{2} = \frac{1}{2}\int_3^{t} u^{-3}\,du$$

$u = 2x+1$

$du = 2\,dx$

$$= \frac{1}{2}\cdot\frac{u^{-2}}{-2} = \frac{-1}{4}\cdot\frac{1}{u^2}\Big|_3^t$$

$$= \frac{-1}{4}\left[\underbrace{\frac{1}{t^2}}_{\to 0} - \frac{1}{9}\right]$$

as $x\to\infty$

$t\to\infty$

Final Ans $\boxed{\frac{1}{36}}$ Convergent.

Jul 2-9:08 AM

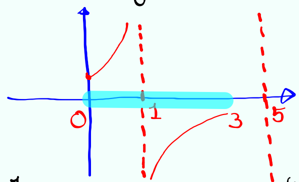
July 2, 2024

Evaluate $\int_0^3 \frac{dx}{x^2-6x+5}$

$x^2-6x+5=0$

$(x-1)(x-5)=0$

disc. at 1 & 5



$$\int_0^1 \frac{1}{x^2-6x+5}\,dx + \int_1^3 \frac{1}{x^2-6x+5}\,dx$$

$$\int_0^1 \frac{1}{x^2-6x+5}\,dx = \lim_{t\to 1^-} \int_0^t \frac{1}{x^2-6x+5}\,dx$$

$\left(\frac{1}{2}\cdot 6\right)^2$

$x^2-6x+5=$

$x^2-6x+9-9+5=$

$(x-3)^2-4$

$$\int \frac{1}{x^2-6x+5}\,dx = \int \frac{1}{(x-3)^2-4}\,dx$$

u^2-2^2

$$=\frac{1}{2\cdot 2}\ln\left|\frac{x-3-2}{x-3+2}\right|$$

$$=\frac{1}{4}\left[\ln|x-5|-\ln|x-1|\right]$$

Now evaluate $\Big|_0^t$

$$=\frac{1}{4}\left[\ln|t-5|-\ln|t-1|-\ln 5\right]$$

Divergent

Now $\lim_{t\to 1^-} \frac{1}{4}\left[\ln\left|\cancelto{\infty}{\frac{t-5}{t-1}}\right|-\ln 5\right]=\infty$

Jul 2-9:17 AM

Find the arc length of the curve

$x=\frac{y^4}{8}+\frac{1}{4y^2}$ for $1\le y\le 2$

$g(y)=\frac{y^4}{8}+\frac{1}{4y^2}$

$$L=\int_1^2 \sqrt{1+[g'(y)]^2}\,dy$$

$g'(y)=\frac{4y^3}{8}-\frac{2}{4y^3}$

$g'(y)=\frac{y^3}{2}-\frac{1}{2y^3}$

$$1+[g'(y)]^2=1+\frac{y^6}{4}-\cancel{2}\cdot\frac{\cancel{y^3}}{\cancel{2}}\cdot\frac{1}{2\cancel{y^3}}+\frac{1}{4y^6}$$

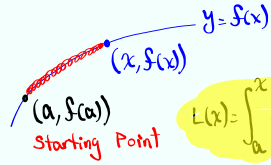
$$=\frac{y^6}{4}+\frac{1}{2}+\frac{1}{4y^6}=\left(\frac{y^3}{2}+\frac{1}{2y^3}\right)^2$$

$$L=\int_1^2 \sqrt{\left(\frac{y^3}{2}+\frac{1}{2y^3}\right)^2}\,dy=\int_1^2\left(\frac{y^3}{2}+\frac{1}{2y^3}\right)dy$$

$$=\frac{1}{2}\int_1^2\left[y^3+y^{-3}\right]dy=$$

Jul 2-9:31 AM

Arc length function



$$L(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

Find the arc length function for

$f(x) = x^2 - \frac{1}{8}\ln x$ starting at $\boxed{x=1}$.

$f'(x) = 2x - \frac{1}{8x}$ $\qquad$ $f'(t) = 2t - \frac{1}{8t}$

$$[f'(t)]^2 = \left(2t - \frac{1}{8t}\right)^2 = 4t^2 - 2\cdot 2t \cdot \frac{1}{8t} + \frac{1}{64t^2}$$

$$= 4t^2 - \frac{1}{2} + \frac{1}{64t^2}$$

$$1 + [f'(t)]^2 = 4t^2 + \frac{1}{2} + \frac{1}{64t^2} = \left(2t + \frac{1}{8t}\right)^2$$

$$L(x) = \int_1^x \sqrt{\left(2t + \frac{1}{8t}\right)^2}\, dt = \int_1^x \left(2t + \frac{1}{8t}\right) dt$$

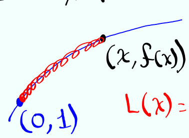
$$= \left(t^2 + \frac{1}{8}\ln t\right)\Big|_1^x = x^2 + \frac{1}{8}\ln x - 1 - 0$$

$$\boxed{L(x) = x^2 + \frac{1}{8}\ln x - 1}$$

Jul 2-9:55 AM

Find arc length function for

$f(x) = \sin^{-1} x + \sqrt{1 - x^2}$ with starting point at $x = 0$



$$L(x) = \int_0^x \sqrt{1 + [f'(t)]^2}\, dt$$

$f(t) = \sin^{-1} t + \sqrt{1 - t^2} \rightarrow (1 - t^2)^{1/2}$

$$f'(t) = \frac{1}{\sqrt{1-t^2}} + \frac{1 \cdot -\cancel{2}t}{\cancel{2}\sqrt{1-t^2}} = \frac{1 - t}{\sqrt{1-t^2}}$$

$$1 + [f'(t)]^2 = 1 + \frac{(1-t)^2}{1 - t^2} = 1 + \frac{1-t}{1+t}$$

$$= \frac{1+t+1-t}{1+t} = \frac{2}{1+t}$$

$$L(x) = \int_0^x \sqrt{\frac{2}{1+t}}\, dt$$

$u = 1+t$ $\qquad$ $t=0$ $\quad$ $u=1$

$du = dt$ $\qquad$ $t=x$ $\quad$ $u=1+x$

$$= \sqrt{2}\int_0^x \frac{1}{\sqrt{1+t}}\, dt$$

$$= \sqrt{2}\int_1^{1+x} \frac{1}{\sqrt{u}}\, du = \sqrt{2}\int_1^{1+x} u^{-1/2}\, du$$

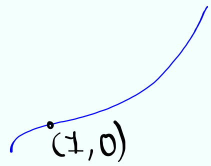
$$= \sqrt{2}\left[\frac{u^{1/2}}{1/2}\right]\Bigg|_1^{1+x} = \boxed{2\sqrt{2}\left[\sqrt{1+x} - 1\right]}$$

Jul 2-10:04 AM

July 2, 2024

Find the arc length function for

$f(x)=\int_1^x \sqrt{t^3-1}\,dt$ with starting point at $x=1$.



$f'(x)=\sqrt{x^3-1}\cdot 1$

$f'(t)=\sqrt{t^3-1}$

$$\sqrt{1+[f'(t)]^2}=\sqrt{1+t^3-1}=\sqrt{t^3}=t^{3/2}$$

$$L(x)=\int_1^x t^{3/2}\,dt=\frac{t^{5/2}}{5/2}\Big|_1^x=\frac{2}{5}\left[x^{5/2}-1\right]$$

$$\boxed{L(x)=\frac{2}{5}\left[x^2\sqrt{x}-1\right]}$$

$[1,4]$ $\quad L(4)=\frac{2}{5}\left[4^2\cdot\sqrt{4}-1\right]$

$$=\frac{2}{5}[31]=\boxed{\frac{62}{5}}$$

Jul 2-10:14 AM

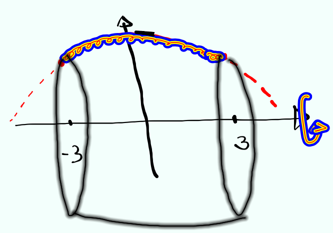
Find the surface area when the curve given by $f(x)=\sqrt{25-x^2}$ is rotated about x-axis from $x=-3$ to $x=3$.

$f(x)=\sqrt{25-x^2}$

$y=\sqrt{25-x^2}$

$y^2=25-x^2$

$x^2+y^2=25$



$$S=\int_{-3}^{3} 2\pi\, f(x)\sqrt{1+[f'(x)]^2}\,dx$$

$$=2\int_0^3 2\pi\sqrt{25-x^2}\sqrt{1+\left[\frac{-x}{y}\right]^2}\,dx$$

$x^2+y^2=25$ $\quad =2\int_0^3 2\pi\sqrt{25-x^2}\sqrt{\frac{y^2}{y^2}+\frac{x^2}{y^2}}\,dx$

$2x+2y\frac{dy}{dx}=0$ $\quad =2\int_0^3 2\pi\sqrt{25-x^2}\cdot\sqrt{\frac{25}{y^2}}\,dx$

$\frac{dy}{dx}=\frac{-x}{y}$

Jul 2-10:22 AM

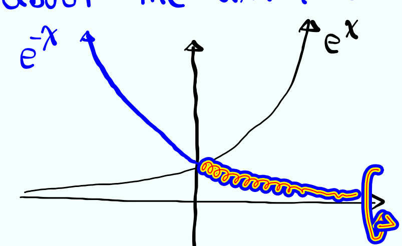
$$= 2\int_0^3 2\pi\sqrt{25-x^2}\cdot\sqrt{\frac{25}{y^2}}\,dx$$

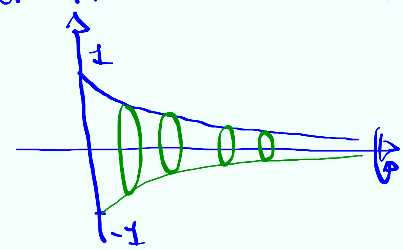
$$= 2\int_0^3 2\pi \cancel{\sqrt{25-x^2}}\cdot\frac{5}{\cancel{y}}\,dx$$

$$= 2\cdot 2\pi\int_0^3 5\,dx = 4\pi\cdot 5x\Big|_0^3 = \boxed{60\pi}$$

Jul 2-10:30 AM

Rotate the infinite curve $f(x)=e^{-x},\ x\geq 0$ about the axis. Find its surface area.





$$S=\int_0^{\infty} 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$$

$$=\lim_{t\to\infty}\int_0^t 2\pi e^{-x}\sqrt{1+[-e^{-x}]^2}\,dx$$

$$=\lim_{t\to\infty}\int_0^t 2\pi e^{-x}\sqrt{1+e^{-2x}}\,dx$$

Jul 2-10:33 AM

$$= \lim_{t\to\infty} \int_0^t 2\pi e^{-x} \sqrt{1+e^{-2x}}\, dx$$

$u = e^{-x}$

$-du = e^{-x}\,dx$

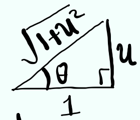
$$= -\lim_{t\to\infty} \int_1^{e^{-t}} 2\pi \sqrt{1+u^2}\, du$$

$x=0 \quad u=1$

$x=t \quad u=e^{-t}$

$$= 2\pi \lim_{t\to\infty} \int_{e^{-t}}^{1} \sqrt{1+u^2}\, du$$

$u = \tan\theta$



$du = \sec^2\theta\, d\theta$

$\int \sqrt{1+u^2}\, du =$

$\int \sqrt{1+\tan^2\theta} \cdot \sec^2\theta\, d\theta =$

$\int \sec^3\theta\, d\theta$

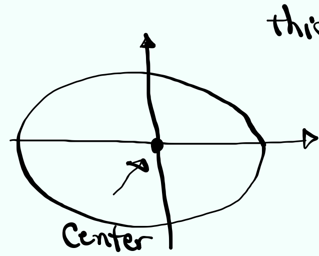
Make Sure to finish this Problem

Jul 2-10:38 AM

Plan for Wednesday

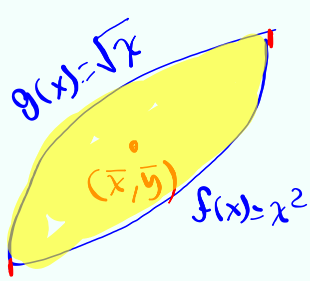
1) QZ at 8:00

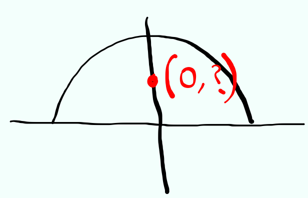
uniform thickness



2) Centroid $(\bar{x}, \bar{y})$

3) QZ at 11:00





Jul 2-10:45 AM

Class QZ 9

Find $\int \frac{2}{x^2-4x+3}\,dx = \int \left[\frac{1}{x-3} - \frac{1}{x-1}\right] dx$

$$\frac{2}{x^2-4x+3} = \frac{2}{(x-3)(x-1)} = \frac{A}{x-3} + \frac{B}{x-1}$$

$= \ln|x-3| - \ln|x-1| + C$

$= \boxed{\ln\left|\frac{x-3}{x-1}\right| + C}$

$2 = A(x-1) + B(x-3)$

$x=1 \rightarrow B=-1$

$x=3 \rightarrow A=1$